

MUTUAL DEBTS COMPENSATION AS GRAPH THEORY APPLICATION

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Key Words:

insolvency, mutual debts, graph theory, creditor — debtor relationship

ABSTRACT.

The article deals with a serious problem of many postcommunist countries i.e. firms' financial insolvency as a result of their mutual debts. Some authors [2], [4], [5] proposed graph theory application in solution of the problem. Creditor – debtor relationship is modelled by a digraf of unpayable debts $G=(V,E)$, where the set of vertices V represents the firms and the set of edges E represents the creditor – debtor relationship. The role of subsidiaries in mutual debts' compensation process is discussed in the article.

1. DIGRAF OF DEBTS

Let $V = \{1, 2, \dots, n\}$ be a set of firms. Then $i \in V$ and $j \in V$ are variables describing firms involved in mutual debt compensation. Debt of the i -th firm towards the j -th firm is denoted as y_{ij} . It is obvious, that $0 \leq y_{ij}$ and that only one of equations $y_{ij} = 0$, $y_{ji} = 0$ is fulfilled.

Let us describe a set of relations as follows

$$(1) \quad H = \left\{ (i, j) \mid y_{ij} > 0 \right\}$$

where

$$(2) \quad \forall h \in H, \quad h = (i, j) \quad y(h) = y_{ij}.$$

Then

$$(3) \quad G = (V, H, y)$$

is the weighted directed graph (digraph) and it will be called the debt relation graph.

Balance of unpayable credits and debts of each firm $i \in V$ can be described as

$$(4) \quad b(i) = \sum_{j=1}^n y_{ji} - \sum_{j=1}^n y_{ij} \quad b^+(i) = \max\{0, b(i)\} \quad b^-(i) = \max\{0, -b(i)\}.$$

The first sum in (4) is the sum of unpayable credits and the second one represents the unpayable debts of the i -th firm.

Note, that

$$(5) \quad \sum_{i=1}^n b(i) = 0$$

because

$$\sum_{i=1}^n b(i) = \sum_{i=1}^n \left(\sum_{j=1}^n y_{ji} - \sum_{j=1}^n y_{ij} \right) = \sum_{i=1}^n \sum_{j=1}^n y_{ji} - \sum_{i=1}^n \sum_{j=1}^n y_{ij} = 0$$

If $v_1 = v_k$, then

$$(6) \quad c = v_1, (v_1, v_2), v_2, \dots, v_{k-1}, (v_{k-1}, v_k), v_k,$$

is a cycle in debt graph $G = (V, H, y)$. Capacity of cycle c is number

$$(7) \quad y(c) = \min_{h \in c} \{y(h)\}$$

If there is a cycle (6) with capacity (7) in graph (3) it means, that there is a cycle of firms, which owe each other in graph. Situation can be simplified by lowering all debts $y(h)$ where $h \in c$, by value $y(c)$

$$y(h) \leftarrow y(h) - y(c).$$

This approach enables removing of minimum evaluated edge and in such a way elimination of the cycle in graph. It means, that creditor – debtor relationship will be removed. It is important to eliminate all cycles in graph. This way we can achieve cyclical debts (and insolvency) elimination in graph. As it is shown in Example 1, the order of cycle elimination is important for achievement of minimal mutual debts in graph.

Let us denote x_{ij} a part of debt y_{ij} , which has been compensated by cycle elimination. The total amount of compensated debts is

$$(8) \quad \sum_{i=1}^n \sum_{j=1}^n x_{ij}$$

The revenue of debts compensation organiser is represented as

$$(9) \quad \sum_{i=1}^n \sum_{j=1}^n c x_{ij}$$

where $c \in (0;1)$ is a portion of compensated debts received by the mutual debts organiser.

The value of x_{ij} must non-negative and less then y_{ij} , which implies that

$$(10) \quad 0 \leq x_{ij} \leq y_{ij}$$

Mutual debts lowering described above must make the balance of debts and credits of any firm in the graph neither better nor worse. It means, that for each firm $v \in V$ must be fulfilled relation

$$(11) \quad \sum_{j=1}^n x_{ji} - \sum_{j=1}^n x_{ij} = 0$$

That means, that the order of cycles' elimination is important for total debts minimising in the graph. The optimal order of cycles elimination and thus achieving maximum debts compensation are given by result of following linear programming model

$$(12) \quad \begin{aligned} & \max \sum_{i=1}^n \sum_{j=1}^n x_{ij} \\ & \sum_{j=1}^n x_{ji} - \sum_{j=1}^n x_{ij} = 0 \text{ for } i = 1, 2, \dots, n \end{aligned}$$

$$0 \leq x_{ij} \leq y_{ij} \text{ for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n.$$

The optimal solution of above mentioned linear programming is equivalent to the solution of the model, that maximizes total revenues of mutual debt compensation organizer and this model takes the following form

$$(13) \quad \begin{aligned} & \max \sum_{i=1}^n \sum_{j=1}^n cx_{ij} \\ & \sum_{j=1}^n x_{ji} - \sum_{j=1}^n x_{ij} = 0 \text{ for } i = 1, 2, \dots, n \end{aligned}$$

$$0 \leq x_{ij} \leq y_{ij} \text{ for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n.$$

2. SUBSIDIARY CENTRE

The situation of mutual debts compensation can be made more appropriate, if there is a subsidiary centre in the graph. Let us have firm $s \notin V$ which organise mutual debts compensation. Let

$$(14) \quad \begin{aligned} \bar{V} &= V \cup \{ n+1 \} \\ \bar{H} &= H \cup \{ (n+1, i) \mid i \in V \} \cup \{ (i, n+1) \mid i \in V \} \end{aligned}$$

Edge $(i, n+1)$ represents fictive debt of the i -th firm towards subsidiary centre and on contrary edge $(n+1, i)$ represents fictive debt of subsidiary centre towards the i -th firm.

Function $\bar{y}(h)$ is defined on the set of edges \bar{H} as follows

$$\begin{aligned}
 \bar{y}(h) &= y(h), \text{ if } h \in H \\
 \bar{y}(h) &= u_{i,s}, \text{ if } h \in \{ (i,s) \mid i \in V \} \\
 \bar{y}(h) &= u_{s,i}, \text{ if } h \in \{ (s,i) \mid i \in V \}
 \end{aligned}
 \tag{15}$$

Values $u_{s,i}$ play role of upper boundary on given subsidiaries towards the i -th firm. Values $u_{i,s}$ have no other importance as to connect the starting and ending vertex of the path in order to create cycle. Because of that the following formula is satisfied

$$u_{s,i} = u_{i,s} \text{ for } i = 1, 2, \dots, n.
 \tag{16}$$

From the point of view of practical interpretation value $u_{s,i}$ plays role of subsidiary centre's cost. Then, linear programming model (13) is justified following

$$\begin{aligned}
 &\max \sum_{i=1}^n \sum_{j=1}^n cx_{ij} - \sum_{j=1}^n x_{sj} \\
 \tag{17} \quad &\sum_{j=1}^n x_{ji} - \sum_{j=1}^n x_{ij} = 0 \text{ for } i = 1, 2, \dots, n \\
 &\sum_{j=1}^n x_{ji} - \sum_{j=1}^n x_{ij} = 0 \text{ for } i = 1, 2, \dots, n \\
 &0 \leq x_{ij} \leq y_{ij} \text{ for } i = 1, 2, \dots, n \text{ and } j = 1, 2, \dots, n
 \end{aligned}$$

where c is return of subsidiary centre obtained from each compensated dollar.

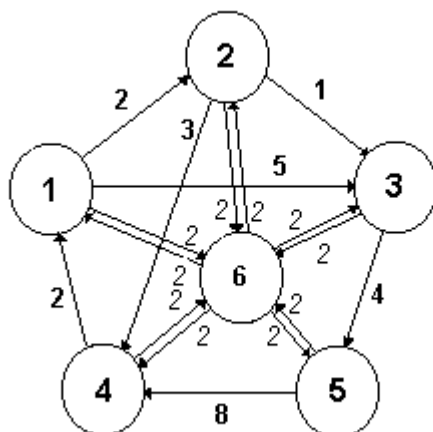


Fig. 1: Debt relations among the firms if optimal order of cycles' elimination is chosen.

Example 1:

The structure of mutual debts among firms $V = (1, 2, 3, 4, 5)$ is represented in graph in Fig. 1. Then, the total amount of mutual debts is 25 mil. USD. Let the coefficient c be 0.5. We assume, that subsidiary centre (vertex 6) has possibility to give subsidiary in amount of 2 mil. USD to each of the firms in the graph. The total amount of given subsidiaries is restricted by amount of 5 mil USD.

Then, the linear programming model maximising income of subsidiary centre is given by following set of equations

$$\begin{aligned}
 &\max \\
 &+ 0,5x_{12} + 0,5x_{13} + 0,5x_{23} + 0,5x_{24} + 0,5x_{35} + 0,5x_{54} + 0,5x_{41} - x_{61} - x_{62} - x_{63} - x_{64} - x_{65} \\
 &-x_{12} \quad -x_{13} \qquad \qquad \qquad +x_{41} \quad +x_{61} - x_{16} \qquad = 0 \\
 &+x_{12} \qquad \quad -x_{23} \quad -x_{24} \qquad \qquad \quad +x_{62} - x_{62} \qquad = 0 \\
 &\qquad \quad +x_{13} \quad +x_{23} \qquad \quad -x_{35} \qquad \qquad \quad +x_{63} - x_{36} \qquad = 0 \\
 &\qquad \qquad \qquad \quad +x_{24} \quad +x_{35} \quad -x_{54} \qquad \quad +x_{64} - x_{46} \qquad = 0 \\
 &+x_{12} \qquad \qquad \qquad \quad +x_{24} \quad -x_{54} \quad -x_{41} \qquad \quad +x_{65} - x_{56} \qquad = 0 \\
 &\qquad \quad +x_{13} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \leq 2 \\
 &\qquad \qquad \quad +x_{23} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \leq 5 \\
 &\qquad \qquad \qquad \quad +x_{24} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \leq 1 \\
 &\qquad \qquad \qquad \qquad \quad +x_{35} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \leq 3 \\
 &\qquad \qquad \qquad \qquad \qquad \quad +x_{54} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \leq 4 \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \quad +x_{41} \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \leq 8 \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad +x_{61}+x_{62}+x_{63}+x_{64}+x_{65} \qquad \leq 2 \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad x_{61},x_{62},x_{63},x_{64},x_{65},x_{16},x_{26},x_{36},x_{46},x_{56} \leq 2 \\
 &\qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad x_{61},x_{62},x_{63},x_{64},x_{65},x_{16},x_{26},x_{36},x_{46},x_{56} \geq 0 \\
 &x_{12}, \quad x_{13}, \quad x_{23}, \quad x_{24}, \quad x_{35}, \quad x_{54}, \quad x_{41} \qquad \qquad \qquad \geq 0
 \end{aligned}$$

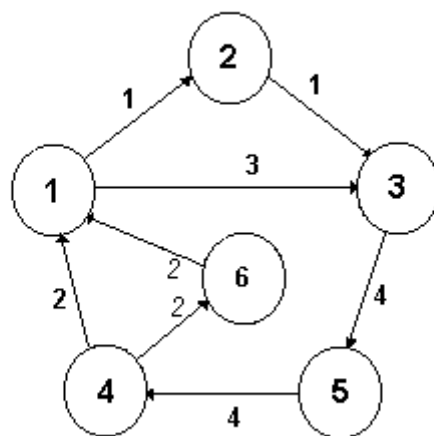


Fig. 2: The graph of compensated debts.

The optimal solution of above mentioned linear programming model is optimal compensation of debts that is given by values x_{ij} represented in graph in Fig. 2. Their sum is 15 mil. UDS.

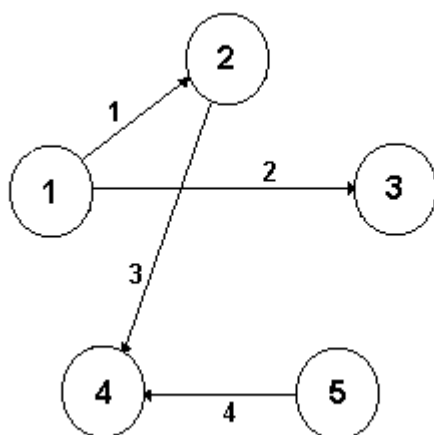


Fig. 3: Residual debts after debts compensation.

The subsidiary centre gives subsidiary to firm 1 in amount of 2 mil. USD.

The residual debts are represented by graph in Fig. 3. It is obvious that final debts' structure contains no cycles. That means we can not diminish total debts amount by further use of algorithms described above. The total residual debts amount 10 mil. USD. The organiser of debts elimination has given subsidiary of 2 mil USD to firm 1 but despite of that he achieves profit of 5 mil USD.

3. CONCLUSION

The above mentioned approach is based on graph theory application in mutual debts compensation. The further development of the approach is possible by involving of bank in the graph and examination of the interest rate change influence on total debts in the graph. Further extension of the models is possible by definition of import and export vertices. Then, the measurement of currency devaluation on total debts' compensation possibilities is appropriate, too.

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